Music as a Branch of Mathematics

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Eduardo J. Calle

Miami Dade College

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Abstract

This presentation explores music through a variety of mathematical lenses including arithmetic, set theory, algebra, probability, statistics, geometry, and calculus. Music composed, arranged, and recorded for this address and featured throughout the presentation will illustrate relationships inherent between music and mathematics including comparisons between Pythagorean and equal-tempered tuning and mathematical analyses of musical elements including loudness, sound waves, and frequencies. Digital recording software was used to analyze data in order to quantify differences in rhythmic feel between musical phrases played by Sonny Rollins and John Coltrane. Additionally, jazz improvisations will generate music as functional set theory.

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Music as a Branch of Mathematics

Musical Mathematicians

During his presidential address regarding mathematics and music to the Mathematical Association of America on September 6th, 1923, Brown University's Raymond Clare Archibald celebrated the ties binding mathematics and music from a historical perspective. From Hermann von Helmholtz's suggestion that math and music share a "hidden bond" visible through the study of acoustics by Joseph Fourier, to the proclamation by Gottfried Leibniz that "music is a hidden exercise in arithmetic, of a mind unconscious of dealing with numbers" (Archibald, 2006, ¶ 3), the history of mathematics is replete with great spirits fascinated and inspired by music. To those in attendance at Vassar College, Archibald listed musical mathematicians including Pythagoras, Pierre-Louis Moreau de Maupertuis, William Herschel, János Bolyia, Augustus De Morgan, Henri Poincaire, Joseph Lagrange, and Albert Einstein to name a few.

During a 1929 interview for The Saturday Evening Post entitled *What Life Means to Einstein*, George Sylvester Viereck writes that physicist, mathematician, and violinist Albert Einstein stated, "If I were not a physicist, I would probably be a musician. I often think in music. I live my daydreams in music. I see my life in terms of music… I get most joy in life out of music" (dos Santos, 2003). Einstein also said that, "Imagination is more important than knowledge. For knowledge is limited to all we now know and understand, while imagination embraces the entire world, and all there ever will be to know and understand" (ThinkExist.com, 2006).

History shows that Einstein was not alone. Fourier, Leonhard Euler, Daniel Bernoulli, and Johannes Kepler contributed immensely to the science of mathematics inspired or challenged by music. Nurtured by earth's atmosphere, supported by mathematical pillars anchored in study, fueled by imagination, necessitated by the need of a creative soul to share unique visions, and realized by a tireless dedication to the celebration of talented passion, music fills the air with sound, minds with wonder, and hearts with joy.

Pythagoras and the Musical Ratios

The Greek philosopher, mathematician, and musician Pythagoras defined the octave as a ratio of 1:2 by discovering that two tones produced on either side of a string bridged in a manner dividing the string into two sections measuring a single unit on one side of the bridge and two units on the other differed in pitch or frequency by one octave (Archibald, 1923). Figure 1 illustrates a string designed to produce one distinct pitch when selecting the segment located on the right side of the bridge and a second pitch sounding one octave higher when plucking, striking, or bowing the segment of the string stretching to the left of the bridge.

Figure 1. An illustration of a musical octave defined by Pythagoras as a ratio of 1:2.

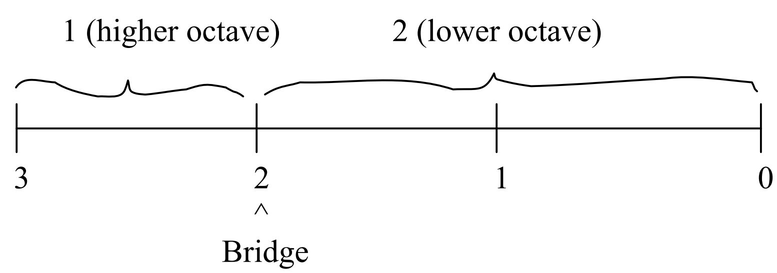


Figure 1. Pythagoras defined the interval of one octave as a ratio of 1:2. The figure depicts a string divided into three equal parts where a bridge demarcates a ratio of 1:2 on the string. Plucking, striking, or bowing the string segment located left of the bridge will produce a tone one octave higher in pitch than the segment located right of the bridge. The numbers illustrate unit lengths with numbers 0 and 3 representing the endpoints of the string.

By plucking, striking, or bowing a single-stringed Greek musical instrument called a monochord depicted in Figure 2, Pythagoras also defined musical intervals of one fifth as a ratio of 2:3 and one fourth as a ratio of 3:4. The methodology applied by Pythagoras to define ratios for the fifth and fourth is illustrated in Figures 3 and 4.

Figure 2. Drawing of a monochord.

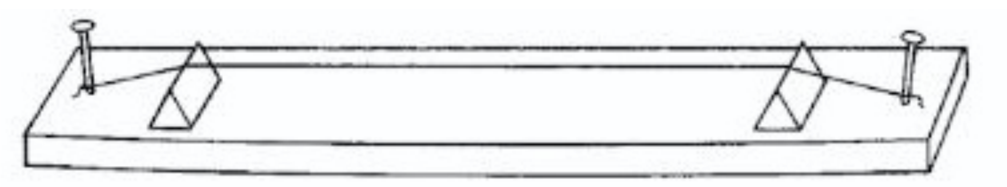


Figure 2. The monochord is a one-string musical instrument whose string is tightly suspended over a soundboard. This drawing of a monochord was obtained from the website: [*http://www.practicalphysics.org/go/Experiment_130.html*](http://www.practicalphysics.org/go/Experiment_130.html)

Figure 3. An illustration of a musical fifth defined by Pythagoras a ratio of 2:3.

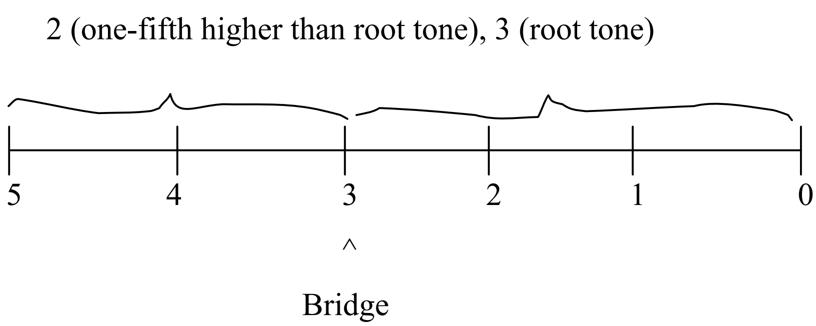


Figure 3. Pythagoras defined the interval of one fifth as a ratio of 2:3. The figure depicts a string divided into five equal parts where a bridge demarcates a ratio of 2:3 on the string. Plucking,

striking, or bowing the string segment located left of the bridge will produce a tone one fifth higher in pitch than the segment located right of the bridge. The numbers illustrate unit lengths with numbers 0 and 5 representing the endpoints of the string.

Figure 4. An illustration of a musical fourth defined by Pythagoras a ratio of 3:4.

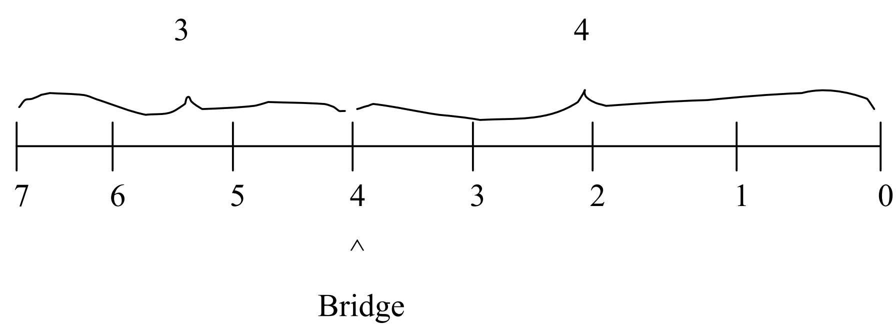


Figure 4. Pythagoras defined the interval of one fourth as a ratio of 3:4. The figure depicts a string divided into seven equal parts where a bridge demarcates a ratio of 3:4 on the string. Plucking, striking, or bowing the string segment located left of the bridge will produce a tone one fourth higher in pitch than the segment located right of the bridge. The numbers illustrate unit lengths with numbers 0 and 7 representing the endpoints of the string.

Based on the findings by Pythagoras illustrated in the preceding section, we can algebraically determine ratios for the remaining tones of the seven-note musical mode called Phrygian by the Greeks and known today as the Dorian mode. Table 1 enumerates the names of the original Greek modes and their corresponding modern counterparts (Frazer, 2006). Currently, the second mode of the C-Major scale is called the D-Dorian mode and contains the notes D-E-F-G-A-B-C in ascending order.

Table 1. Greek modes names and musical notation (C-Major)

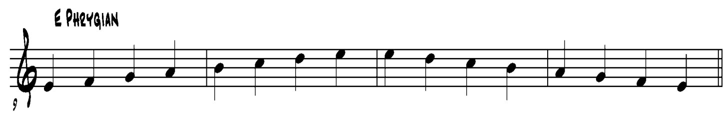
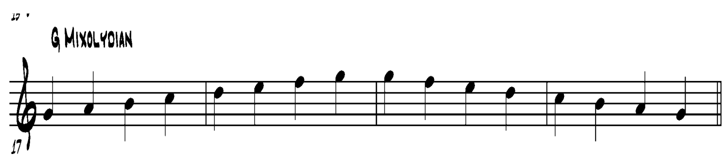
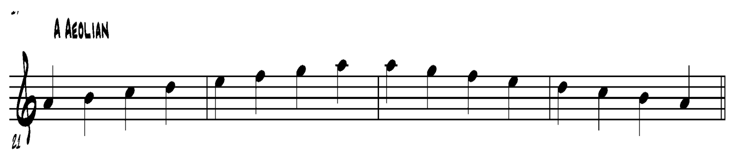
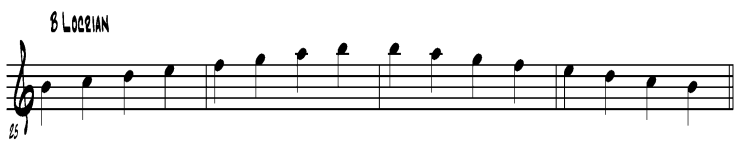
Greek name	Modern name	Music notation
Lydian	Ionian	
Phrygian	Dorian	
Dorian	Phrygian	
Hypolydian	Lydian	
Hypophrygian	Mixolydian	
Hypodorian	Aeolian	
Mixolydian	Locrian	

Table 1. The left most column of the table lists the original Greek names for each mode of the major scale. The remaining columns list the modern names and corresponding musical notation for each of the seven modes.

By applying the ratios provided by Pythagoras, it is possible to determine ratios for other fourths and fifths contained within the Dorian mode. Based on ratios for the octave of 1:2 and the fifth of 2:3, we can determine that the ratio for the note E2 sounding one-fifth higher than A (2/3) is 4:9 (see Figure 4). In order to physically transpose the newly created fifth down one octave so that it sounds within the range between notes D1 and D2 one would need to double the length of the string producing the newly formed ratio and corresponding tone. Algebraically, such a transposition can be expressed in the form $(2/3)(2/3)(2) = 8/9$ or the ratio for the note E2. The ratio for B, the sixth note of the D-Dorian mode, can subsequently be determined by applying a ratio of 2/3 to the value of E1, $(8/9)(2/3) = 16/27$. The ratio for the note C2 can be found by finding an interval one fourth higher than G or $(3/4)(3/4) = 9/16$ and the ratio for the note F can be found by generating a ratio for a note one fifth lower than C2 (9:16). The ratio for a note F sounding one fifth lower than C2 (9:16) can be found by multiplying the ratio for C2 (9:16) by the inverse ratio of one fifth or the ratio 3:2. Consequently, we can generate the ratio for F1 by calculating the expression $(9/16)(3/2) = 27/32$.

Figure 5. Pythagorean ratios for the notes defining the D-Dorian mode.

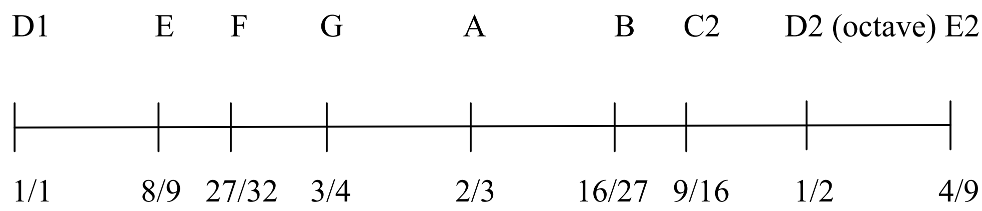


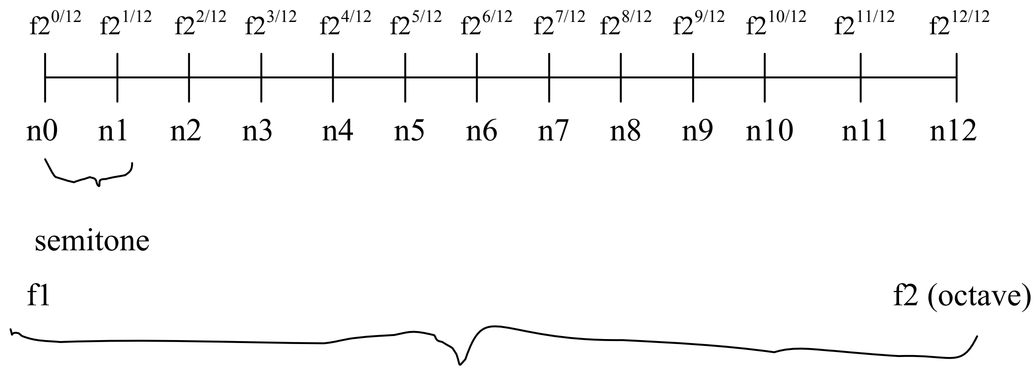
Figure 5. The figure details the computed ratios of the D-Dorian mode based on the findings of Pythagoras. Since a fifth is produced by a string length ratio of 2:3, consecutive fifths are produced by a ratio of 4:9. Once having computed a ratio for consecutive fifths the user can find ratios for the remaining elements by finding ratios for fourths and fifths sounding above and below each newly computed ratio.

Applying a different algebraic approach we can compute the ratio for a whole step or distance between the fourth (G) and fifth (A) mode degrees in Figure 5 by finding a value for a ratio "x" so that the ratio 3:4 multiplied by a number "x" produces the ratio 2:3. Solving the equation $3/4x = 2/3$ generates the ratio of 8/9 or 8:9 for a whole step. Applying this constant value for a whole step, we can generate the ratio for E, the second note displayed in Figure 4 by solving the expression $D1x = E$ or $(1/1)(8/9) = 8/9$. Inversely, since the distance between the fourth (G) and the third (F) of the Dorian mode is also a whole step, we can find a ratio value for F such that $F(8/9) = 3/4$ or 27/32. We subsequently discover the ratio for the sixth note of the mode (B) is 16/27 by multiplying the ratio 2/3 representing the fifth (A) by 8/9. Lastly, the ratio generating the seventh note (C) can be found by solving the expression $C(8/9) = 1/2$ or 9/16.

Equal-Tempered Tuning

The piano keyboard features an 88-note keyboard that divides each octave into twelve semitones or half steps. Based on Pythagoras discovery that the ratio of one octave is 1:2, we can begin to define the semitone by first expressing the relationship between two tones separated by one octave. A ratio of 1:2 between note (n) and its octave can be expressed as n:2n. Exponentially, a note n can be expressed as (1 * n) or (2^0 * n) while 2n can be rewritten as (2^1 * n). In order to divide one octave into twelve equal semitones as detailed in Figure 6, we can divide the distance between (n * 2^0) and (n * 2^1) into twelve equal segments written (n * $2^{0/12}$), (n * $2^{1/12}$), (n * $2^{2/12}$), (n * $2^{3/12}$), (n * $2^{4/12}$), (n * $2^{5/12}$), (n * $2^{6/12}$), (n * $2^{7/12}$), (n * $2^{8/12}$), (n * $2^{9/12}$), (n * $2^{10/12}$), (n * $2^{11/12}$), and (n * $2^{12/12}$) respectively. The resulting frequencies, dividing the octave into twelve equidistant semitones, produce a chromatic scale and define the most widely used Western music tuning system called equal-tempered tuning. Additionally, the distance between adjacent semitones is divided into one hundred units called cents.

Figure 6. A graphic interpretation of one octave divided into twelve semitones.



A depiction of one octave divided into twelve semitones.

Figure 6. The graphic diagrams twelve semitones equally dividing one octave ranging from semitone or frequency f to semitone f2. The distance between semitones is defined as $2^{n/12}$ *where n represents the number corresponding to the position of the note in an ascending chromatic (12-tone) scale. Note that* $2^0 = 1$ *and* $2^0{:}2^1$ *is equivalent to 1:2 defining the Pythagorean ratio for the octave.*

Standardized Tuning

Slonimsky (2001) credits French acoustician Joseph Sauveur (1653–1716) as the first person to calculate the number of vibrations by a specific pitch and also offered the first scientific explanation for upper partials of a fundamental tone called overtones. Tennenbaum (1991) states that Sauveur quantified that the note “do” or middle C oscillated 256 times per second while studying tones created by organ pipes and vibrating strings. Sauveur’s collected writings, published by the French Academy of Science from 1700 to 1713 (Rasch, 2006), precede professor Rudolph Hertz’ (1847–1894) first transmission and reception of radio waves and measurements of the velocity and wavelength of electromagnetic waves by approximately 175 years (Jenkins, 2006). The number of completed cycles per second by a “periodic phenomenon" (Institute for Telecommunication Sciences, 2006) such as a sound wave is defined as one hertz (Hz) in honor or professor Hertz. Consequently, Sauveur's measurement of middle C at 256 cycles per second translates to 256 Hz.

During the 1940s a global movement to standardize tuning led to the general adoption of 440 Hz as the corresponding frequency for the note "A" or "la" sounding one sixth above middle C or "do". Based on Sauveur's findings, the current tuning standard is somewhat higher than the tuning reference ranging between 427 and 430 Hz commonly used in Europe by composers such as Bach, Mozart, and Beethoven.

MIDI Note Numbers and Frequencies

In addition to music technology contributions including the development of the Prophet 5 and Prophet 600 synthesizers, the American music technology company Sequential Circuits introduced a paper at the 1981 Audio Engineering Society (AES) convention proposing a new digital interface called the Universal Synthesizer Interface (USI) (Akins, 2007). In 1982, a consortium of manufacturers reviewed, tweaked, and accepted the work by Sequential Circuits developers as the standard electronic communication interface and called it Musical Instrument Digital Interface (MIDI). MIDI has become a standard and indispensable creative tool for manufacturers and musicians facilitating communication between computers, synthesizers, and a wide range of electronic devices since the introduction of built-in MIDI interfaces on the Sequential Circuits Prophet 600 in 1981 and other popular synthesizers such as the Yamaha DX-7 synthesizer in 1982.

Though a MIDI controller can feature as many as 128 keys, pads, or buttons capable of triggering a maximum 128 notes defined as MIDI note number integers 0 - 127, a full-sized MIDI synthesizer keyboard features eighty-eight keys. As detailed by Valenti (1988), MIDI coding designates a specific number to each key on a standard keyboard ranging from $A_0 = 21$ on the left to $C_8 = 108$ on the right. Note that the MIDI note numbers available on a controller featuring 128 keys or input sources will range from key $C_{-1} = 0$ at the extreme left (lowest pitch) to $G_9 = 127$ at the extreme right (highest pitch). Though MIDI allows the user great flexibility with respect to tuning, transposition, and a wide range of expressive parameters, the standard setting for all MIDI keyboards assigns note number 60 to C_4 (middle C). When depressed, each key on the MIDI keyboard produces a frequency corresponding to a note on the musical staff.

Finding the corresponding equal-tempered frequency for a note on a MIDI keyboard can be achieved by using MIDI note number 69 corresponding to A4 on the keyboard. In this instance we will choose MIDI note number 69 as a constant or reference because the frequency corresponding to this note is the current tuning standard 440 Hz. It is important to note that producers, composers, conductors, orchestras, and musicians in a variety of musical settings and instances sometimes choose a slightly higher standard tuning frequency such as 444 Hz.

Given a frequency value for A4 such as 440 Hz, a corresponding MIDI note number (69), and discovering that an equal-tempered semitone "n" ranging between MIDI note 0 and 127 can be expressed as $2^{n/12}$ allows us to find the frequency for a MIDI note number (n – 69) positions away from A4 (69) by computing the math expression ($440 * 2^{(n-69)/12}$). Since the equal-tempered octave is divided in increments of twelve and Pythagoras established that the ratio for an octave is 1:2, we can verify our findings by calculating the frequency values corresponding to notes any number of octaves lower or higher than the reference value for n (69). Note numbers corresponding to various octaves of n can be found by adding or subtracting multiples of twelve. For example, in order to find a MIDI note number one octave higher than 69 we simply compute 69 + 12 = 81 and plug the new note number into the expression this way; frequency in Hz of

MIDI note number 81 = 440 * $2^{(81-69)/12}$. The solution for this expression renders a frequency value for MIDI note number 81 (A5) yields 880 Hz (440 * $2^{12/12}$ = 440 * 2 = 880 Hz) in accordance with the 1:2 ratio defining the octave. Similarly, MIDI note number 93 produces a frequency of 1760 Hz (440 * $2^{(93-69)/12}$ = 440 * $2^{24/12}$ = 440 * 2^2 = 1760). The frequency ratios corresponding to MIDI note numbers 69 (440 Hz) and 81 (880 Hz), and MIDI note numbers 69 (440 Hz) and 93 (880 Hz) are 1:2 and 1:4 respectively concurs the findings by Pythagoras. The MIDI note number for a note A2 sounding two octaves (2 * 12) lower than A4 can be calculated by first solving the expression 69 – 24 = 45. Therefore, the frequency in Hz of note number 45 is 110 Hz (440 * $2^{(45-69)/12}$ = 440 * $2^{-24/12}$ = 440 * 2^{-2} = 440/4 = 110 Hz). Once again, the ratio of 4:1 between the frequencies corresponding to A_4 (440 Hz) and A_2 (110 Hz) respectively is in accordance with Pythagorean principles.

Musicians, students, and other parties interested in finding the corresponding MIDI note number for a given frequency can use logarithms to simplify the mathematic expression, f (frequency in Hz) = 440 * $2^{(n-69)/12}$. The expression can be simplified as detailed in Figure 7.

Figure 7. Method for finding the corresponding MIDI note number for a given frequency

Original expression	$f = 440 * 2^{(n-69)/12}$
Simplification step 1	$f/440 = 2^{(n-69)/12}$
Simplification step 2	$\log_2 (f/440) = (n-69)/12$
Simplification step 3	$12 * \log_2 (f/440) = n - 69$
Formula for finding a MIDI note number given the frequency (Hz) of the MIDI note	$n = (12 * \log_2 (f/440)) + 69$

Figure 7. Given the frequency (f) for a note in Hz, it is possible to find the corresponding MIDI note number represented by the variable "n" using the formula detailed and simplified in the figure.

Applying the formula detailed in Figure 7, we can verify the earlier mentioned assertion that the note A above middle C used by Bach, Mozart, and Beethoven between the late 17th century and early 19th centuries was somewhat lower than the reference A4 (440 Hz) commonly used today. Sauveur defined the frequency of middle C as 256 Hz. Applying the MIDI note number corresponding to middle C (60) and substituting a reference frequency of 256 Hz yields the expression $Hz = 256 * 2^{(n-60)/12}$. Since the MIDI note number corresponding to A above middle C is 69, we can find the frequency (f) for A above middle C typically used in Europe by solving the expression $f = 256 * 2^{(69-60)/12}$. We can therefore verify that traditional European tuning was lower (A4 = 430.54 Hz) than modern tuning (A4 = 440 Hz) by solving the expression using a calculator or by inserting the function [*=256*(POWER(2,(69-60)/12))*] into a Microsoft Excel spreadsheet.

Macintosh users interested in finding MIDI note numbers for given frequencies and vice versa without having to physically compute the data can download a free note to frequency calculator widget courtesy of Jacklin Studios from the website located at http://www.jacklinstudios.com/software/notefreq/.

Differences Between Pythagorean and Equal-Tempered Tuning

Though it is possible that Greek musicians may have been able to recall a fixed pitch from memory or used available reference tones such as those produced by bells, pipes, or environmental sources as a starting point, the ratios discovered by Pythagoras enabled musicians of the day to produce music without the need for a reference tuning-note. A comparison of frequencies values based on Pythagorean and equal-tempered tuning ratios illustrates differences between both tuning systems and highlights the fact that intervals sounded by ancient Greek instruments and those produced by instruments built to function in an equal-tempered tuning system sound slightly different.

Figure 8. The A major pentatonic scale.

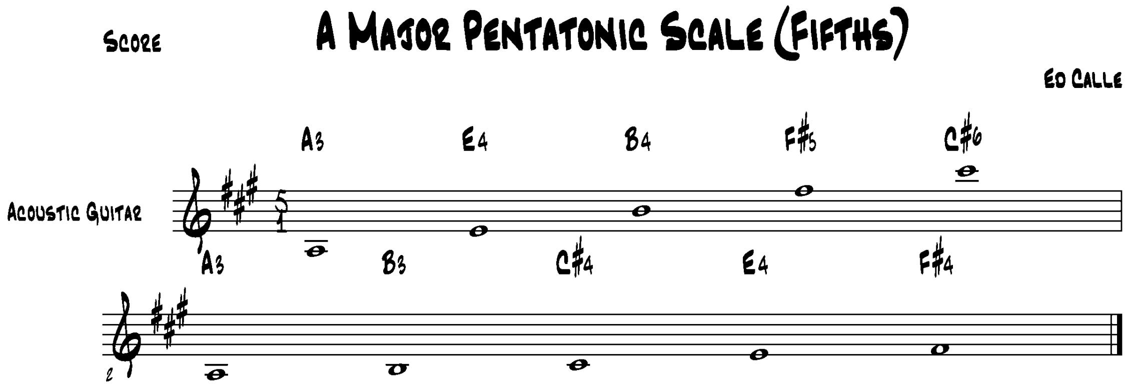


Figure 8. The figure displays musical notation for notes constructing the A major pentatonic scale in sequential perfect fifths and ascending stepwise order within one octave respectively. Standard MIDI note names specifying note placement on a piano keyboard are displayed above each note.

Figure 8 displays a musical staff containing five notes each separated by a perfect fifth. These notes also represent the elements of the A major pentatonic scale as detailed in the second measure of Figure 8. Using a reference frequency of 440 Hz for the note A4, Table 2 lists the corresponding frequency values for each note displayed in Figure 8 computed by applying Pythagorean ratios and the formula for equal-tempered tuning (see page 13).

Table 2. Comparison of frequencies for the A-major pentatonic scale based on Pythagorean and equal-tempered relationships.

Reference Freq. (Hz)	Note name	MIDI note #	Pythagorean ratio (top)	Pythagorean ratio (bottom)	Pythagorean freq. (Hz)	Equal-tempered freq. (Hz)
440.00	A4	69	1	1	440.00	440.00
	A3	57	1	2	220.00	220.00
	E4	64	3	2	330.00	329.63
	B4	71	3	2	495.00	493.88
	F#5	78	3	2	742.50	739.99
	C#6	85	3	2	1113.75	1108.73
	A3	57	1	2	220.00	220.00
	B3 (from A3)	59	9	8	247.50	246.94
	C#4 (from F#4)	61	3	4	278.44	277.18
	E4 (from B3)	64	8	9	330.00	329.63
	F#4 (from B3)	66	3	2	371.25	369.99

Table 2. The data displayed in the table details frequencies corresponding to the notes displayed in Figure X based on Pythagorean ratios and equal-tempered tuning.

Math Around Writer's Block

A dreaded and frustrating state of the creative psyche for a composer, writer, or improvising musician is sometimes termed writer's block, a slump, a funk, or a dry-spell. When encountered, any type of block can delay or prevent the productive output of creative, talented, and dedicated artists by placing an array of cognitive and emotional roadblocks in the way of the creative process ranging from confusion, anxiety and frustration to creative paralysis and self-doubt. Musicians encountering such a block might employ a mathematical lens focused on probability to uncover a myriad of new themes, variations, and permutations capable of serving as alternate routes to renewed creativity.

During a recent conversation regarding this presentation, Dr. Bernard Geltzer and the author queried the feasibility of determining the number of possible variations or permutations for a song. Possible answers to the original question emerged from our discussion triggering a startling reminder of the sheer magnitude of possible musical combinations available to the creative community.

Exploring the possible combinations of a twelve-tone row, defined as a series of twelve non-repeating tones situated within a one-octave chromatic scale, produces a combination of twelve possible distinct pitches in the first position, eleven possible selections in the second position, ten possible notes in the third position decreasing in similar fashion down to one remaining pitch in the twelfth position. In mathematical terms, this twelve-tone sequence can be expressed as twelve-factorial (12! = 12 * 11 * 10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 *1) generating 479,001,600 possible non-repeating sequences of twelve chromatic tones. Admittedly, most popular or commercially successful music is not based on such twelve-tone sequences. Nonetheless, combinations numbering more than 479 million offer talented and interested parties a mammoth creative palette.

When exploring a more commercial music approach, computing the possible combination of seven non-repeating tones confined within a one-octave major scale generates 5,040 combinations (7!). If we change the parameters to include seven notes diatonic to a specific major scale without restricting the number of times a note may repeat the number of combinations grows to 7^7 or 823,543 combinations. It is not beyond the scope of reason to imagine that at least one of these combinations may yield inspiration for a new creative work. And yes, that number is still based on restricting note choices to a single octave. This is an important point since many songs such as America's National Anthem cover a range exceeding one octave. In fact, the traditional rendition of *The Star Spangled Banner* ranges one octave plus one fifth explaining why it is often a difficult song to perform for singers with limited ranges. Note that singers such as Whitney Houston and Mariah Carey, and instrumentalists including Arturo Sandoval and the author typically perform two-octave renditions of *The Star Spangled Banner*.

Figure 9. The five modes of the A major pentatonic scale.

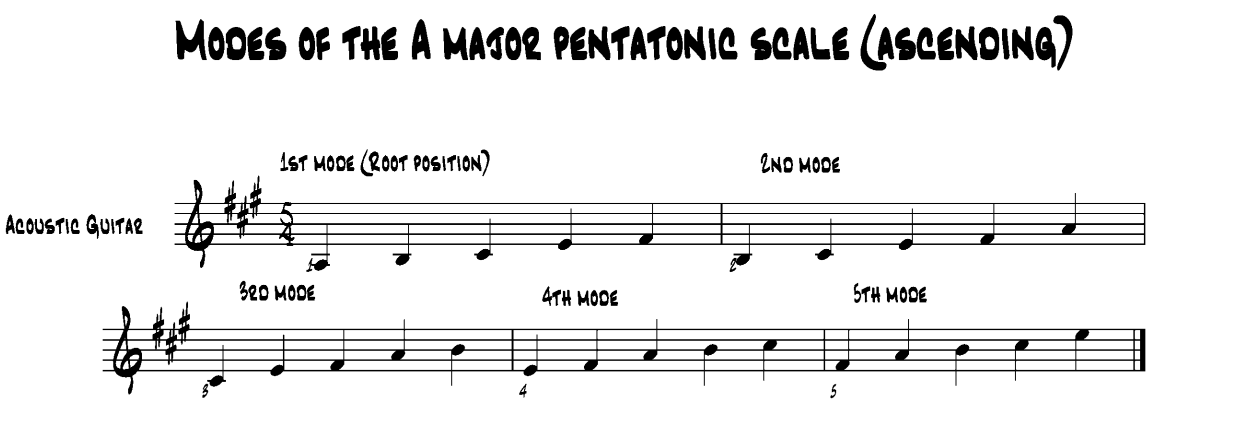


Figure 9. The figure displays music notation for the five modes of the A major pentatonic scale. Note that each mode represents a new sequential ordering of the notes.

Students of jazz improvisation spend time learning to identify, construct, play and apply pentatonic scales and their derivative structures in a variety of musical situations. Though many amateur and professional musicians limit their musical vocabulary to include twelve major pentatonic scales each based on one of the 12 major keys (see Figure 9), the number of available scales containing five unique pitches within one octave produces 95,040 (12 * 11 * 10 * 9 * 8) scale combinations, each generating five modes or orderings beginning on consecutive scale degrees of the parent pentatonic scale (see Figure 20). In total, the number of pentatonic scales and their sequential orderings (modes) produces 475,200 (5 * 95,040) pentatonic combinations that can function as vehicles for composition, arranging, orchestration, improvisation, or exercises applied towards the development of technical proficiency. Anyone facing a creative musical block may take comfort knowing that the set of musical possibilities based on pentatonic scales numbers more than 475,000 combinations without introducing repeated notes.

Figure 10. Traditional rendition of Twinkle, Twinkle Little Star

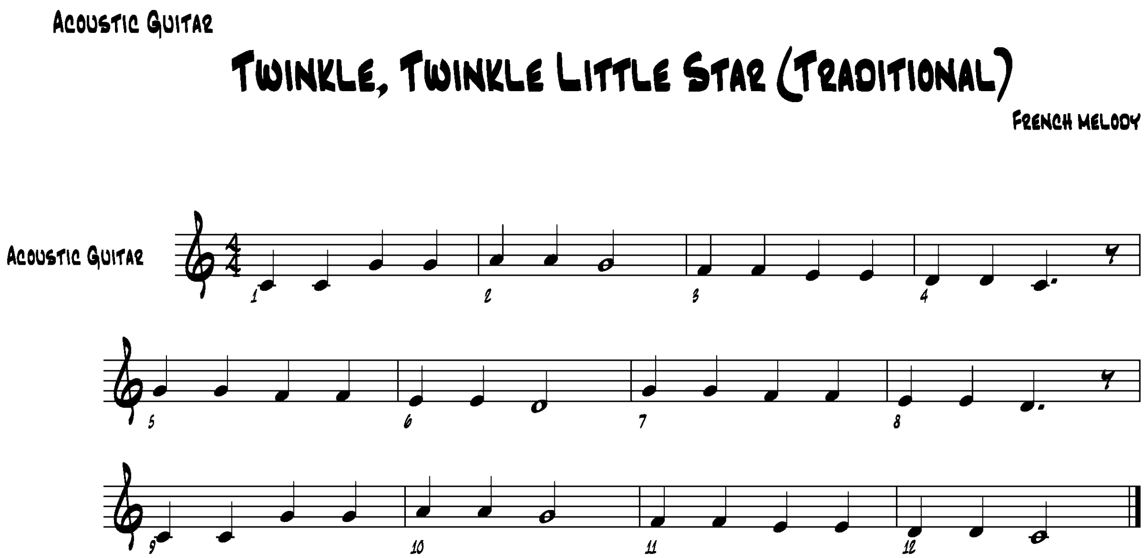


Figure 10. The figure shows musical notation for the song Twinkle, Twinkle Little Star in the key of C Major.

The version of the well-known French melody *Twinkle, Twinkle Little Star* displayed in Figure 10 contains forty-two notes and two rests. How many permutations of this version of the song are possible considering that all orderings follow the exact rhythmic pattern detailed in Figure 10? If we restrict the note choices for new permutations to be contained within a one-octave C major scale ranging from C4 to B4 and disallowing substitutions of notes for rests or vice versa, each note could be substituted by one of the seven notes contained in the C major scale and each rest remains unchanged. Possible permutations given these parameters number 17,294,405 ($7^{42} + 2$). Figure 11 illustrates two such possible variations.

Figure 11. Two permutations based on Twinkle, Twinkle Little Star

Figure 11. The figure presents music notation for two possible permutations based on the song Twinkle, Twinkle Little Star. The permutations apply a maximum of seven rhythmically equivalent substitutions for each note or rest in the song generating 17,294,405 (7^{42} + 2) possible permutations based on the original version illustrated in Figure 7.

Sound Wave Properties and Formulas

Huber and Runstein (2005) point out that the velocity of a sound wave is temperature dependent. Consequently, the speed of a sound wave traveling "through the air at 68ºF (20ºC) is approximately 1130 feet per second (ft/sec)" (p. 38) and increases by 1.1 ft/sec with each 1ºF increase in temperature. As discussed, acoustic frequency measured in Hertz is defined as the number of complete cycles of a wave propagating during one second. Thus, a 60 Hz frequency (f = 60 Hz) completes 60 cycles per second while one cycle or period occurs every 1/60th of a second (period = 1/f). The wavelength (λ) can be expressed as the ratio between the velocity (v) and frequency (f) of the sound wave ($\lambda = v / f$). Therefore, a 60 Hz frequency sounding at a temperature of 68ºF yields a wave of length of approximately 18.83 feet.

French mathematician Jean Baptiste Joseph Fourier is quoted as saying, "Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them" (University of St. Andrews Scotland, 2006). Fourier's theorem states that, "any complex waveform is the sum of sinusoids" (Jayne, 2003). Sine waves, also called sinusoids, are periodic functions of the form: y = a sin b(x – c) + d or y = a cos b(x – c) + d (Hirsch & Schoen, 1985, p. 158). Because sound waves produced by acoustic musical instruments and vocalists are complex waveforms, Fourier's theorem provides a visionary tool for the study and development of synthesized sound, electronic music, and audio engineering. Figure 12 illustrates mathematical expressions for elements of a sine function including amplitude, period, phase shift, and vertical shift. Figure 12 is based on the work detailed by Hirsch and Schoen (1985, p. 158).

Figure 12. The sine function

Sine function: y = a sin b(x – c) + d or y = a cos b(x – c) + d

Amplitude (musically referred to as volume): |a|

Period (one complete excursion of a sound wave): 2π/|b|

Phase shift: c units to the right if c > 0 or |c| units to the left if c < 0

Vertical shift: d units up if d > 0 or |d| units down if d < 0

Figure 12. The figure details the mathematically elements and properties associated with the sine function. Movement along the axis mentioned in the figure refers to the x-axis and y-axis of the xy graph.

Loudness

Sound results when a vibrating body generates longitudinal waves that propagate through mediums such as air and water causing atmospheric disturbances. The healthy human ear perceives miniscule atmospheric disturbances as sound. In fact, Huber and Runstein (2005) point out that one-microbar equals $1x10^{-6}$ (one-millionth) of standard atmospheric pressure and state that most people can hear atmospheric disturbances measuring 0.0002 microbar representing a change of $2x10^{-10}$ (20-billionths) in normal atmospheric pressure. In addition to being able to detect miniscule differences in atmospheric pressure, the range of human hearing extends from 20 Hz to 20,000 Hz (20k Hz).

One dyne (dyn) is defined as the energy required to force the acceleration of a mass weighing one gram (g) by one centimeter (cm) per second squared (sec^2) (http://www.thefreedictionary.com/dyne, 2006). Huber and Runstein (2005) define sound-pressure level (SPL) as the amount of acoustic pressure "built up within an atmospheric area", typically one square centimeter (cm^2). When measured in dyne, sound-pressure levels are expressed as the ratio dyn/cm^2.

One decibel (dB), named after Alexander Graham Bell and meaning "1/10th of a bell" (Huber & Runstein, 2005, p. 52), represents a logarithmic value quantifying intensity differences between two energy levels including SPL, voltage (v), and wattage (w). The threshold of hearing, defined as the "minimum sound pressure that produces the phenomenon of hearing in most people" (Huber and Runstein, p. 137), is defined as the sound-pressure level reference (SPL_{ref}) measured at 0 dB equivalent to the previously described change in atmospheric pressure measuring $2x10^{-10}$ microbars. Huber and Runstein, Pierce (2006), and a host of sources detail the formula for computing an SPL rating in dB as: dB SPL = 20 log SPL/SPL_{ref}, where SPL_{ref} = 0.0002 dyn/cm^2. Since decibels measure the difference in intensity between two sources, computing the intensity difference between a sound level (sl) measuring 100 dyn/cm^2 and a reference level (rl) measuring .01 dyn/cm^2 can be accomplished by solving the expression, dB SPL = 20 log sl/rl, as demonstrated in Figure 13.

Figure 13. Comparison of sound level between a residence and an airline cabin.

Let the reference level (rl) = 0.01 dyn/cm^2 and the sound level (sl) = 100 dyn/cm^2.

Solving for the dB intensity difference between these two sound sources we find:

dB intensity difference SPL = 20 log (sl/rl)

dB intensity difference in SPL = 20 log (100/0.01)

dB intensity difference in SPL = 20 [log 100 – log 0.01]

dB intensity difference in SPL = 20 [2 – (-2)]

dB intensity difference in SPL = 20 [4]

dB intensity difference in SPL = 80 dB

Figure 13. The figure illustrate a step by step method for finding the intensity difference measured in decibels (dB) between two sound sources whose levels are given in dyn/cm^2.

Alternatively, Vanderheiden (2006) asserts that a 20 dB difference in SPL represents a tenfold change in sound pressure. Using rules for logarithms to solve the expression 40 dB = 20 log x, we can describe the intensity associated with a 40 dB difference in sound level between an average residence, defined as 50 dB by Huber and Runstein (2005, p. 54), and a sound level of 90 dB produced inside the cabin of a commercial airplane (The Engineering ToolBox, 2005). Given the expression $y = \log_b x$ is equivalent to $b^y = x$ (Umbarger, 2006, p.6), we discover that a 40 dB SPL increase is equivalent to a hundred-fold change in SPL by solving the equation 40 dB = 20 log x (see Figure 14). Verifying the data displayed in Figure 13 we show that the 10,000-

fold change in intensity between the given reference level (rl = 0.01) and sound level (sl = 100) is equivalent to the solution of the logarithmic expression 80 dB = 20 log x (x = 104).

Figure 14. Change in sound level intensity resulting from an increase of 40 dB SPL.

SPL increased by 40 dB

The variable x represents an x-fold change in level intensity.

$40 \text{ dB} = 20 \log x$

$2 = \log x$

$x = 10^2 = 100$

Figure 14. The figure details a step-by-step method for finding the change in intensity given the amount of change in decibels.

The Geometry of Chords and Scales

Chords, defined as "a combination of three or more pitches sounded simultaneously" (http://www.answers.com/topic/chord-1, 2006), offer a harmonic framework for musicians performing musical ideas in an improvised fashion. The relationship between chords and scales can be expressed through a geometric lens. Musical notation specifies changes in pitch in a vertical manner and rhythm or time in a horizontal manner. Thusly, three or more notes sounding at a singular moment in time are notated vertically. Studying the notated F7 altered chord in the first measure of the musical staff in Figure 15 illustrates the chord as a vertical structure. In contrast, the F# melodic minor scale constructed from the notes defining the F7 altered chord displayed in measures two through nine of the staff in Figure 15 illustrates the horizontal nature of a scale. Therefore, musicians can conceptualize chords as vertical structures represented or defined by horizontal structures called scales when applying a geometric approach to improvisation.

Figure 15. Musical notation for the F7 altered chord and its corresponding scale.

Figure 15. The figure illustrates the vertical nature of an F7 altered chord and the horizontal nature of the scale constructed from the tones of the chord. A scale formed by a set of chord tones is often referred to as a chord-scale. The chord-scale corresponding to an F7 altered chord is the ascending mode of the F# melodic minor scale.

Set Theory and Jazz Improvisation

As discussed, the equal-tempered octave is divided into twelve semitones. Using set notation, the elements (notes) of the C chromatic scale set (C-chromatic) can be expressed as C-chromatic = {C, C#, D, Eb, E, F, F#, G, G#, A, Bb, B} (See Figure 16). The C Dorian scale (C-

Dorian) can be expressed as a set containing seven elements (notes); C-Dorian = {C, D, Eb, F, G, A, Bb}. The set C-Dorian is a proper subset (⊂) of the set C-chromatic because every element (∈) in C-Dorian is also an element of the set C-chromatic but C-chromatic contains elements that are not elements (∉) of C-Dorian. Therefore, C-Dorian ⊂ C-chromatic and {C#, E, F#, G#, B} ∉ C-Dorian. The elements of the set {C#, E, F#, G#, B} define the fifth mode of the E major pentatonic scale (E-pentatonic = {E, F#, G#, B, C#}). Since the elements of set E-pentatonic are not elements of the set C-Dorian, E-pentatonic is the complement of set C-Dorian. Applying this information in a musical context provides a musician entrenched in a C Dorian tonality a set of notes with a strong and recognizable structure that will create harmonic and melodic tension when applied. Applying set theory in a musical context, E major pentatonic represents the harmonic structure most dissonant with C Dorian thus providing a creative tool useful when "stretching" harmony or forcing one tonality onto another.

Figure 16. The C chromatic, C Dorian, and E major pentatonic scales.

Figure 16. The figure shows musical notation for the C chromatic, C Dorian, and E major pentatonic scales in ascending fashion.

Often when improvising, musicians playing single-note melodies strive to approximate or define a specific chord structure with a few notes. Set theory provides a useful method for achieving such an objective. Figure 17 displays musical notation for the C minor thirteenth chord set (Cmin13), the corresponding C Dorian (C-Dorian) chord-scale set, three major pentatonic scale subsets of C-Dorian, and five additional three-note subsets of C-Dorian. Using set notation to express relationships detailed in Figure 17 we notice that C-Dorian is a subset of ($\subseteq$) Cmin13. The Bb pentatonic set $\subset$ C Dorian and the union ($\bigcup$) of the Eb and F pentatonic sets produces the C Dorian scale set (Eb pentatonic $\bigcup$ F pentatonic = C Dorian). The intersection ($\bigcap$) of set C (1-2-5) = {C, D, G} and Eb (1-2-5) = {Eb, F, Bb}, expressed C (1-2-5) $\bigcap$ Eb (1-2-5), renders the empty set ($\varnothing$) meaning that the sets have no elements in common. Musicians wanting to avoid note redundancy may benefit from the study and application of harmonic or scalar combinations of subsets whose intersections produce the empty set. Contrastingly, subsets containing common elements offer musicians tones common to those subsets while still affording the musician alternative note choices.

Figure 17. Music notation for subsets of the C minor thirteenth chord (Cmin13).

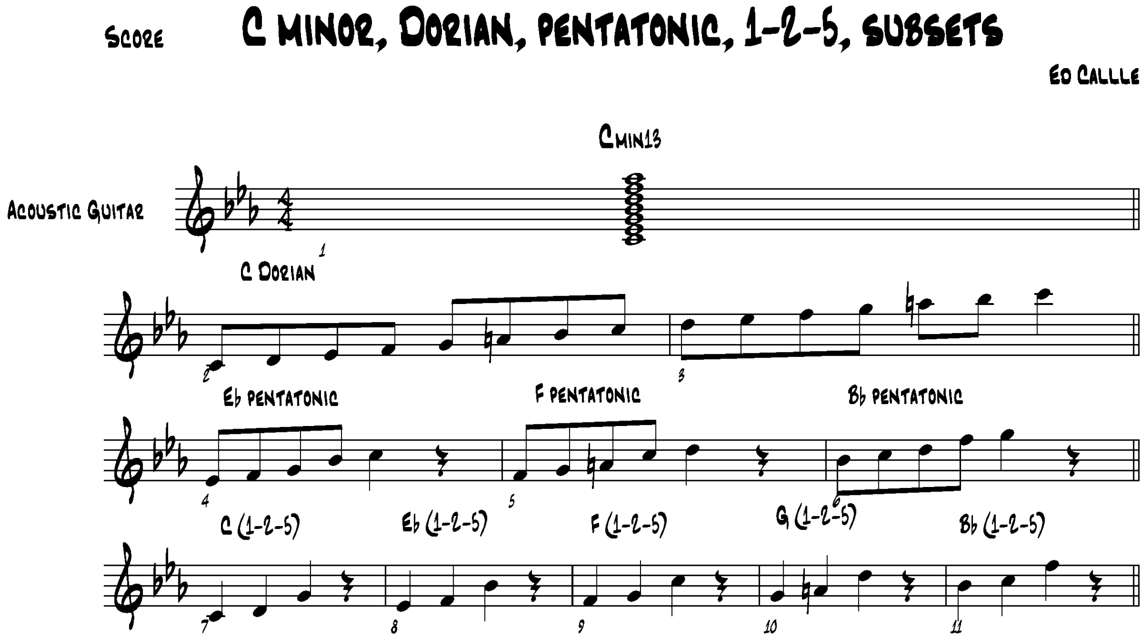


Figure 17. The figure displays music notation for a C minor thirteenth chord, the C Dorian scale, the Eb, F, and Bb major pentatonic scales, and five sets of three note series diatonic to C Dorian comprised of major seconds and a perfect fourth (1-2-5).

Music as Mathematics

Music notation specifies a precise occurrence in time of a distinct harmonic or melodic event or sequence of events. Simultaneously, music displayed on a staff can also detail changes in frequency (pitch), amplitude (volume), style, length, and speed over a prescribed period of time quantified in measures (bars) and beats. Rhythm, or event occurrence and sequence, is notated horizontally from left to right while pitch changes are notated vertically on a variety of staff systems typically containing five lines, each separated by one of four spaces. Musicians learn to translate information shown outside the staff through recognition of additional lines called ledger lines (see Figure 18). As do most initial staves or staves containing a variance in time signature of a musical work, the staff visible in Figure 18 also displays time signatures defining the number of beats (pulses) occurring in each bar and the type of rhythmic unit (note) equaling one beat. Time signatures are specified as positive integers or ratios greater than one (> 1) while counting number multiples of two define the default rhythmic value receiving or equaling one beat per measure. Mathematically speaking, musicians count or compute numeric data while interpreting and performing simultaneous and often frequent changes along related vertical and horizontal axis (see Figure 19).

Figure 18. Musical staff.

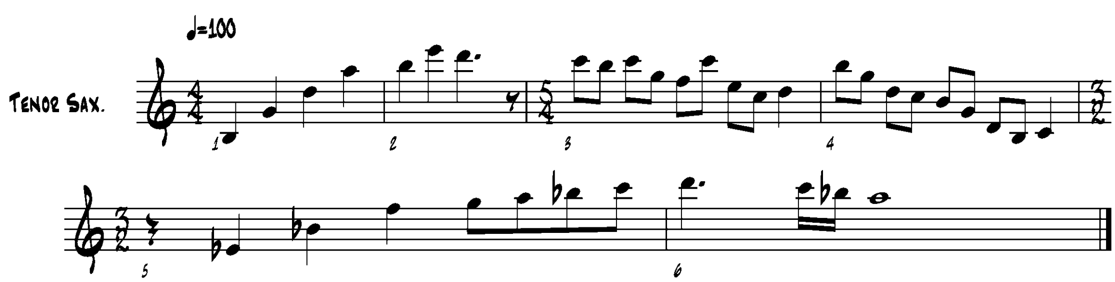


Figure 18. The figure displays a musical staff containing various time signatures, notes within the staff, and notes located outside the staff defined by tangent or embedded ledger lines. The suggested tempo is one hundred beats per minute notated by the marking quarter note equals one hundred.

Figure 19. Musical staff with superimposed imaginary x and y-axis.

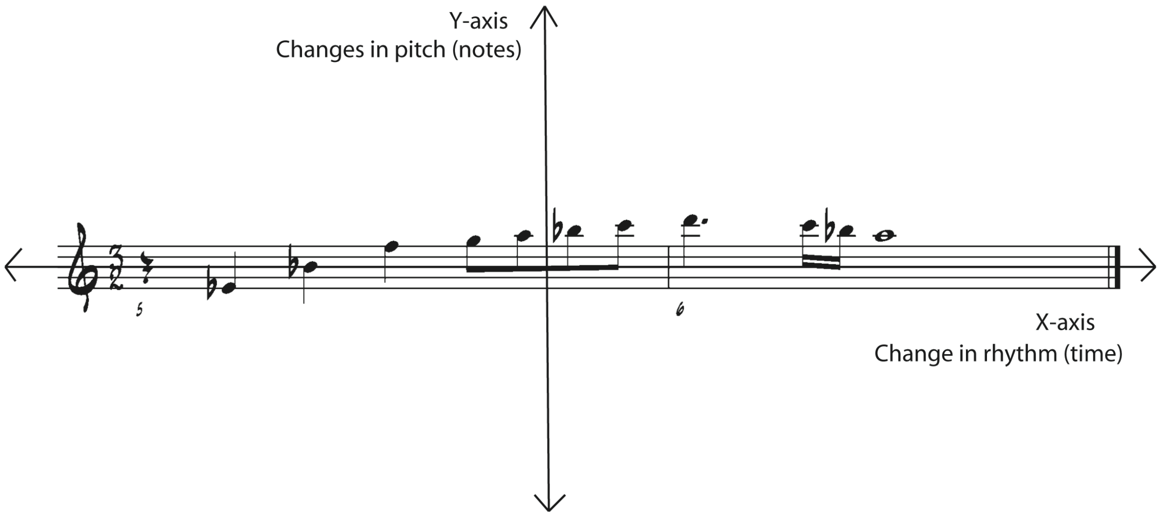
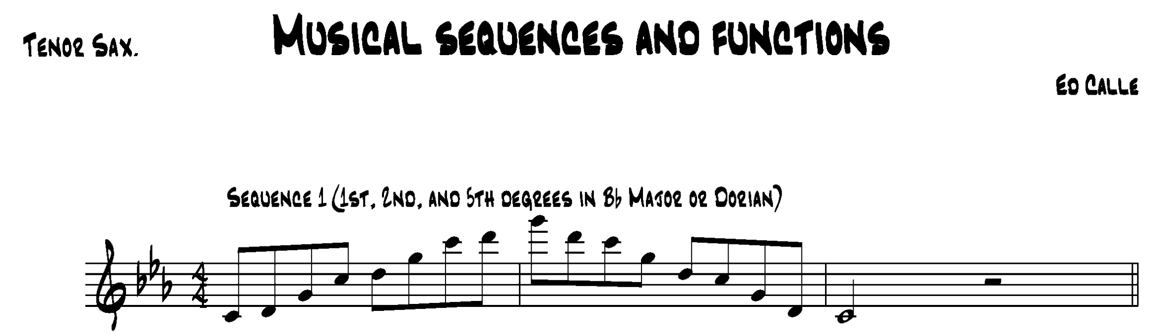


Figure 19. The figure illustrates a musical staff with superimposed imaginary x and y-axis demarcating changes in pitch as vertical events along an imaginary y-axis and changes in rhythm or time as horizontal events along an imaginary x-axis.

In order to study musical data in a mathematical fashion one can convert musical rhythms into numbers or coordinates along an x-axis. For this discussion, the author will use Microsoft Excel to generate numerical values by defining the measure, beat, and note position of each musical event. Though the process can be tedious, the precise location of a recorded musical event can be defined using digital recording software such as Digidesign's Pro Tools. Digital recording and MIDI software divide one beat into ticks. For the purpose of this discussion, each beat will be divided into 960 ticks. Based on this information one can generate a numerical value for rhythmic events by adding the values for the measure number, the beat number, and tick position. In order to identify the location of a musical event occurring at measure one, beat one, and tick one of a measure containing four beats as 1.0 we can use the formula: measure number +(((beat number – 1) + (tick position/(4*960))).

Notes or pitches can be easily converted to numbers and consequent points on the y-axis by using their corresponding MIDI note numbers. Applying this technique enables us to convert musical notation (see Figure 20) to numerical data and input the data into Microsoft Excel in order to generate graphs and even equations for the general trend lines of the graphs. Though beyond the scope of this presentation, mathematicians proficient in calculus and differential equations can generate functions representing such graphs. Future study is needed to determine the musical consequences resulting from derivatives of music-based functions.

Figure 20. Musical notation converted to numerical data and a corresponding graph.



Tenor Note	Piano note	Note number	Y coordinate	Measure	Beat	Tick	X coordinate
C4	Bb3	46	-14	1	1	0	1.000
D4	C3	48	-12	1	1	480	1.125
G4	F3	53	-7	1	2	0	1.250
C5	Bb4	58	-2	1	2	480	1.375
D5	C4	60	0	1	3	0	1.500
G5	F 4	65	5	1	3	480	1.625
C6	Bb5	70	10	1	4	0	1.750
D6	C5	72	12	1	4	480	1.875
G6	F5	77	17	2	1	0	2.000
D6	C5	72	12	2	1	480	2.125
C6	Bb4	70	10	2	2	0	2.250
G5	F4	65	5	2	2	480	2.375
D5	C4	60	0	2	3	0	2.500
C5	Bb4	58	-2	2	3	480	2.625
G4	F3	53	-7	2	4	0	2.750
D4	C3	48	-12	2	4	480	2.875
C4	Bb2	46	-14	3	1	0	3.000

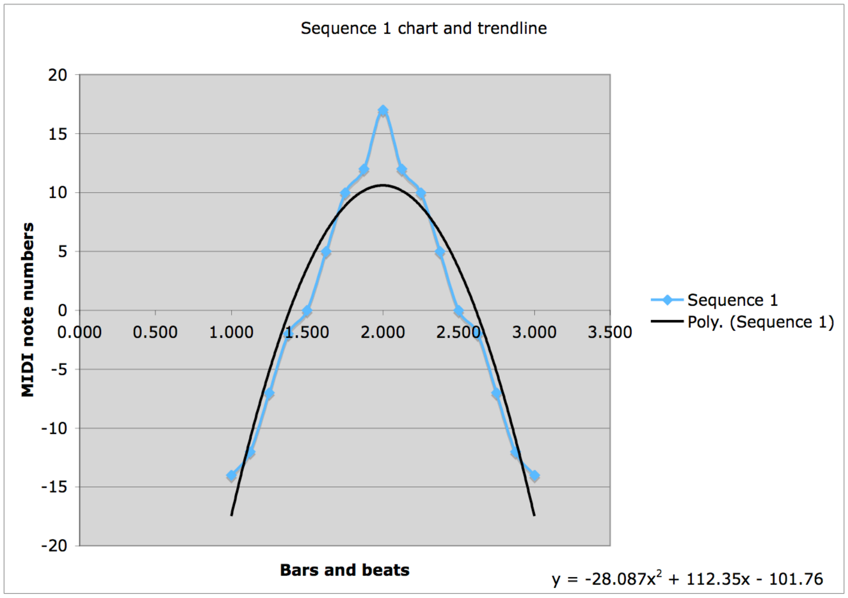


Figure 20. *The figure shows musical notation for a sequence of notes based on the Dorian scale. The subsequent table details the process designed in Microsoft Excel to convert the musical data into numbers and coordinates on the xy-graph. The data is then graphed and a trend line and corresponding function is calculated and graphed using Excel's chart function.*

Math and Musical Feel

Quantization is a process used by MIDI programmers to adjust the rhythmic placement or feel of a musical event. Computer software such as Digidesign's Pro Tools, Mark of the Unicorn's Digital Performer, and Propellerhead's ReCycle allow musicians to manipulate both MIDI data and recorded audio in order to achieve a more precise performance of their creative vision. As a saxophonist and student of jazz one becomes keenly aware of differences and similarities in feel, style, and note choice between jazz musicians. Researching giants of jazz enables us to enjoy, examine, and analyze their contributions with intent to synthesize and apply this newly gained information in a unique manner. As a doctoral student one recognizes a similarity between empirical research and jazz improvisation. In fact, it is this colleague's observation that jazz researchers (musicians) encounter a high degree of difficulty presenting their findings because their works must be delivered to audiences in an improvised fashion.

John Coltrane and Sonny Rollins are jazz giants who can be heard trading four-measure musical ideas on the Sonny Rollins Quarter 1956 Prestige Records release entitled *Tenor Madness*. In jazz lingo, an exchange of musical ideas alternating every four measures is called *trading fours*. Figures 21 and 22 display transcriptions of Coltrane and Rollins trading fours while improvising on Sonny Rollins' blues composition *Tenor Madness*.

Figure 21. Musical notation of transcribed musical phrases by John Coltrane.

Figure 21. The notation displays the musical notation for a series of four-measure improvised phrases performed by John Coltrane during a recording with Sonny Rollins.

Figure 22. Musical notation of transcribed musical phrases by Sonny Rollins.

Figure 22. The notation displays the musical notation for a series of four-measure improvised phrases performed by Sonny Rollins during a recording with John Coltrane.

Converting the recorded data into numbers using Pro Tools and Excel one finds that both Coltrane and Rollins placed their chosen notes in varying rhythmic positions. Table 3 shows the numerical data corresponding to the phrase played by Coltrane in measures 11-13 (Figure 21) and the answering phrase played by Rollins in bars 13-15 (Figure 22). These two phrases were chosen because of the similarity of note choices by the saxophonists and represent an example of call and response dialogue common during trading of musical ideas in jazz improvisation.

Table 3. Numerical representation of musical phrases performed by John Coltrane and Sonny Rollins.

John Coltrane phrases measures 11-13 (Figure 21)	MIDI note number	Y-axis coordinate	Measure number	Beat number (based on 4 measures per bar)	Ticks position (based on 960 ticks per beat)	X-axis coordinate
C4	58	-2	11	1	67	11.02
C4	58	-2	11	1	952	11.25
E4	62	2	11	2	541	11.39
D4	60	0	11	3	173	11.55
F4	63	3	11	3	599	11.66
A4	67	7	11	4	42	11.76
C5	70	10	11	4	572	11.90
E4	62	2	12	1	38	12.01
G4	65	5	12	1	704	12.18
A4	67	7	12	1	943	12.25
C5	70	10	12	2	648	12.42
Bb4	68	8	12	4	574	12.90
A4	67	7	12	4	769	12.95
G4	65	5	13	1	164	13.04
Sonny Rollins phrases measures 11-13 (Figure 21)	MIDI note number	Y-axis coordinate	Measure number	Beat number (based on 4 measures per bar)	Ticks position (based on 960 ticks per beat)	X-axis coordinate
C4	58	-2	13	1	182	13.05
G3	53	-7	13	1	681	13.18
C4	58	-2	13	2	138	13.29
E4	62	2	13	2	654	13.42
C4	58	-2	13	3	142	13.54
D4	60	0	13	3	691	13.68
E4	62	2	13	3	874	13.73
G4	65	5	13	4	674	13.93
F4	63	3	14	1	86	14.02
C4	58	-2	14	1	746	14.19
F4	63	3	14	2	140	14.29
C4	58	-2	14	2	591	14.40
F#4	64	4	14	3	19	14.50
G4	65	5	14	3	667	14.67

A4	67	7	14	3	940	14.74
C5	70	10	14	4	560	14.90
D5	72	12	15	4	765	15.95
E5	74	14	15	1	197	15.05
C5	70	10	15	1	829	15.22

Table 3. The table details the note choices and placements of musical phrases improvised by John Coltrane and Sonny Rollins on the Impulse release entitled Tenor Madness. The music was transcribed and then analyzed by the author using Digidesign's Pro Tools software.

Inspecting the data we notice that Coltrane and Rollins place notes on similar beats of a bar in very different locations. If we divide each beat exactly in half, the first note would occur at tick zero while the second note would occur at tick 480. In this example, John Coltrane plays the note on beat one of measures 11 and 12 at tick positions 67 and 38 respectively. Sonny Rollins positions the notes performed on beat one of bars 13 and 14 on ticks 182 and 86 respectively. Though each musician varies the rhythmic placement of the note in random fashion, the general trend is for Coltrane to place his notes closer to the beginning of the beat than Rollins. In fact, Coltrane places the second note of his phrase on beat one, tick 952 slightly in advance of the second beat (see Table 3). Using jazz terminology, we could say that Sonny *lays back* while Coltrane plays more *on top* throughout the course of this musical dialogue. A possible cause for this tendency is that Rollins is responding to a statement by Coltrane. Further study should be conducted in order to better assess the general tendencies and causal effects inherent when reacting to a musical statement in an improvised manner. Additionally, the sample data is insufficient for purposes of a general trend regarding rhythmic placement by the musicians. A rigorous study of Coltrane's and Rollins' rhythmic tendencies may provide an interesting topic for extensive future research but is beyond the scope of this discussion.

The process used to analyze the information began by importing the commercially obtained audio recording available on compact disc (CD) into Pro Tools using the import audio feature. Since the original master is a mixed and mastered two-track recording, it is difficult to isolate individual performances by musicians whose primary responsibility is generating and maintaining a steady tempo such as the bassist or drummer. Working to maintain a steady tempo in a traditional jazz setting, bassist Paul Chambers is playing one note at the start of each beat (http://www.allaboutjazz.com/php/article.php?id=23731). Using Chambers' performance as the guide for defining the distance between beats in order to calculate a tempo and produce a grid dividing the performance into quantifiable units requires isolation or enhancement of the bass frequencies. The bass frequency range produced by this recording extends from 40 Hz to well over 200 Hz. For purposes of this project, the author isolated the bass response of the original recording by applying a low-shelf equalizer (EQ) to enhance frequencies from 60 Hz to 150 Hz while simultaneously reducing all other frequencies. The equalized signal was then amplified and recorded onto a separate Pro Tools track. Using the newly created isolated bass track, a tempo map and grid were created by finding the start of each bass waveform (note) and calculating the tempo for each beat using the Pro Tools Beat Detective feature. Jazz musicians rarely record using a reference click track or tempo guide and thus tend to generate a variable tempo. In fact, rhythm section instruments such as the bass and drums often anchor the tempo helping to hold the ensemble together. Based on this knowledge, the author generated a reference tempo guide in order to study note placement by Coltrane and Rollins.

The subsequent findings detailed in this paper are for educational purposes only and in no way reflect an opinion about the quality of the performances by John Coltrane or Sonny Rollins. In this colleague's opinion, these men are musical giants whose contributions serve as a guiding example of tangible musical results generated when creative talent, study, dedicated energy, and application are synthesized. Mathematics, as applied in this example, provides insight while celebrating the complexity and beauty of the creative mind.

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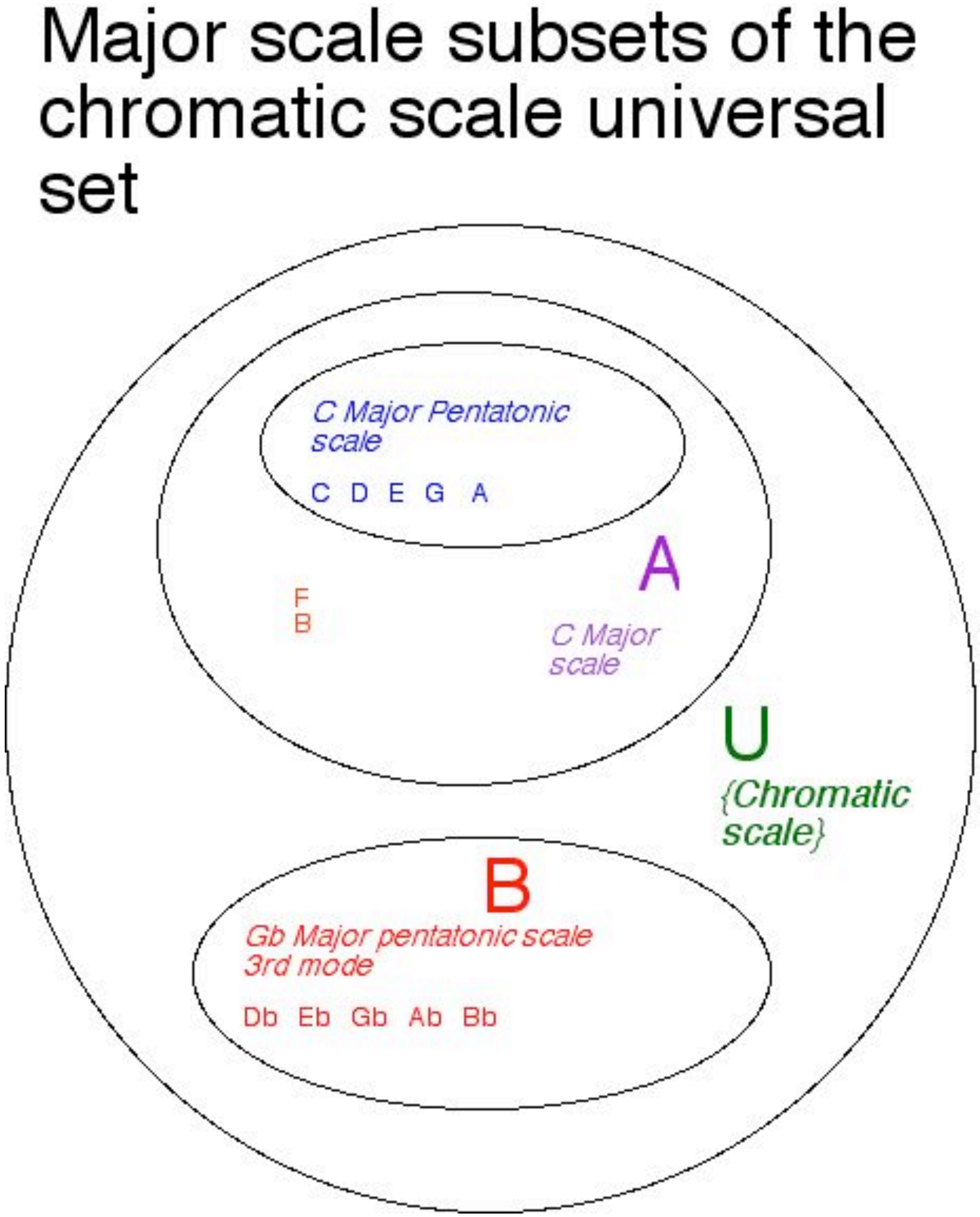
Appendix A

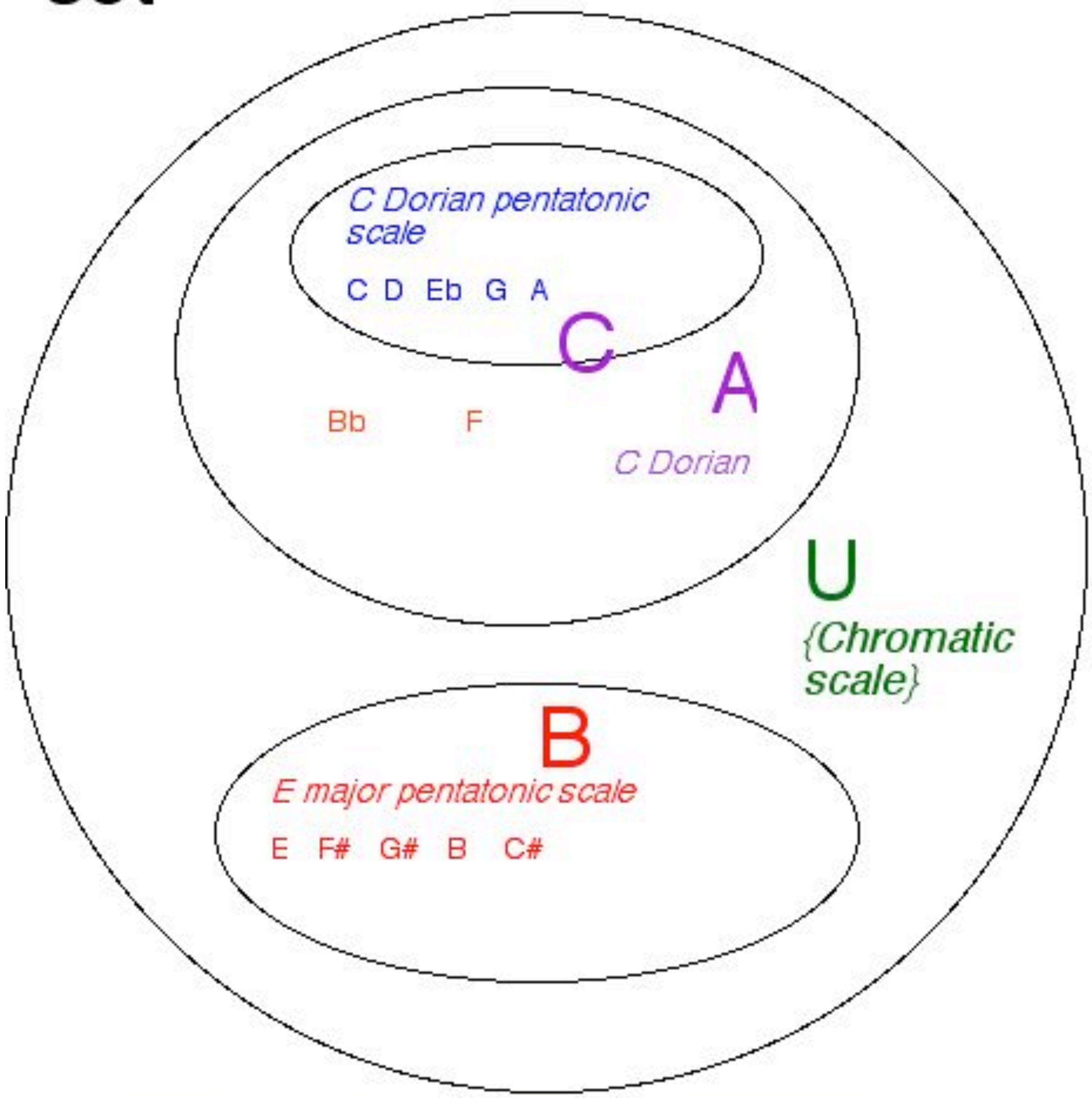
Major Scale Subsets



Appendix B

Dorian Mode Subsets

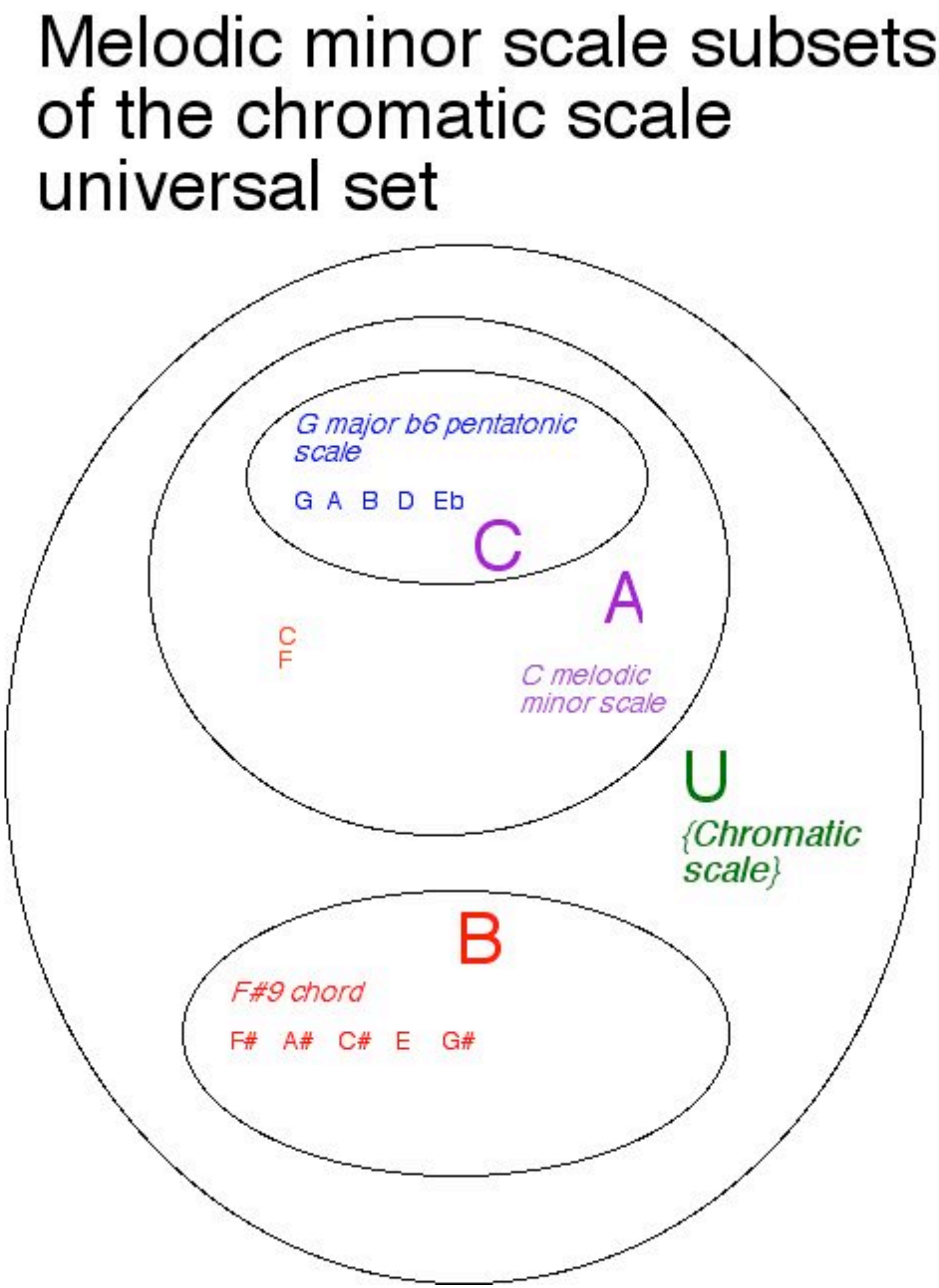
Dorian mode subsets of the chromatic scale universal set



C is a subset of A. A and B are subsets of U. B is ta complement of C and A.

Appendix C

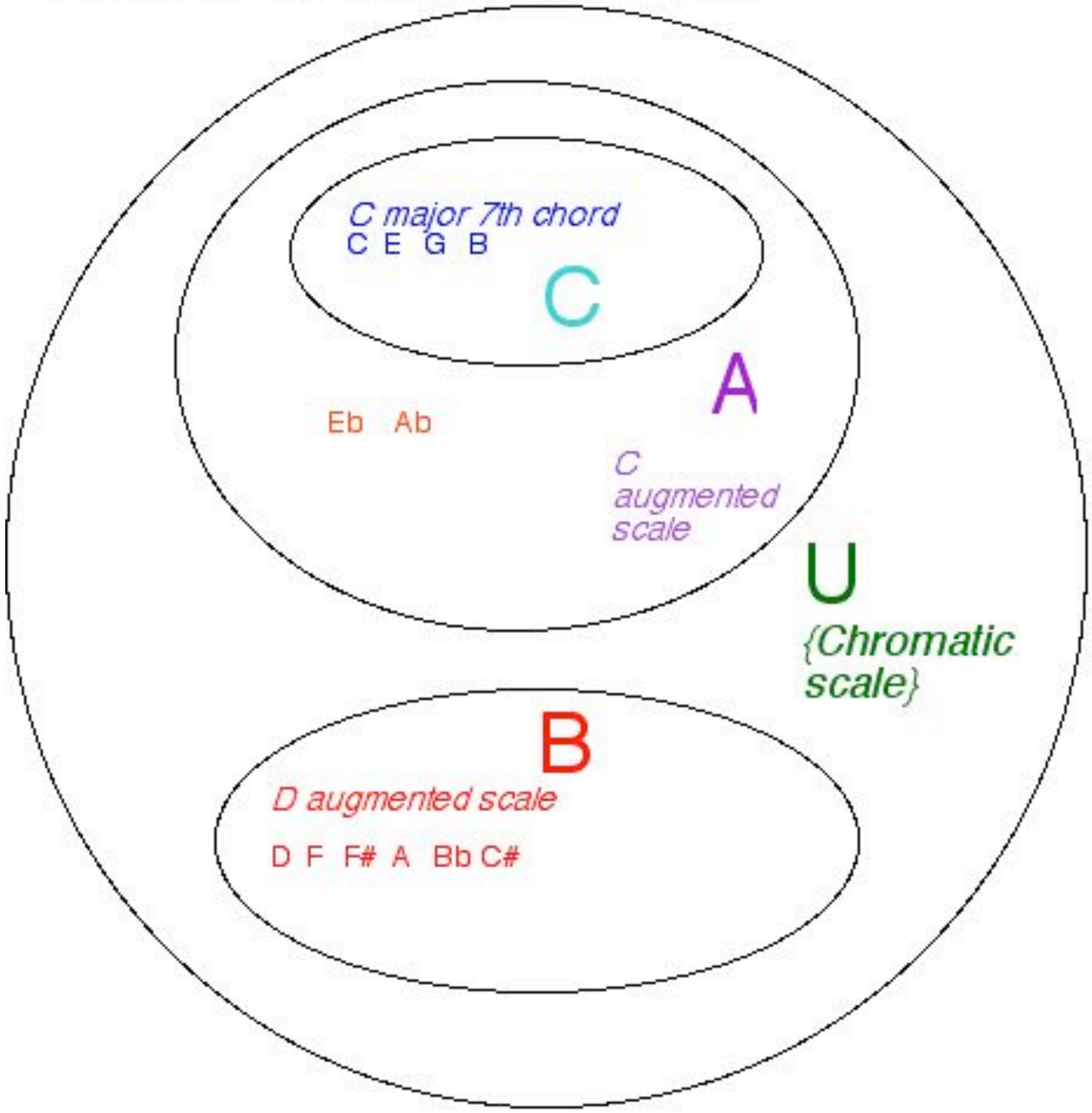
Melodic Minor Scale Subsets



Appendix D

Augmented Scale Subsets

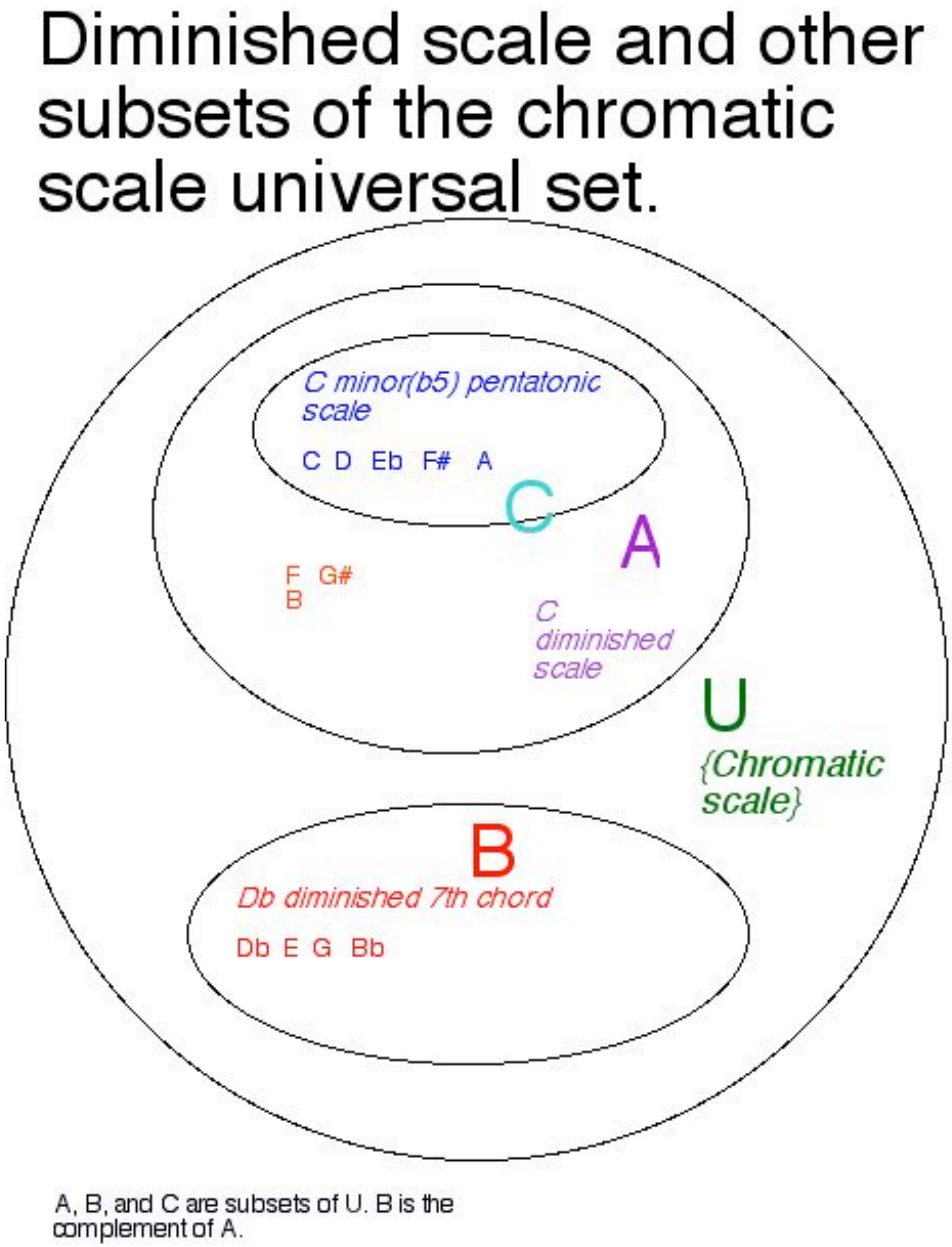
Augmented scale and other subsets of the chromatic scale universal set.



A, B, and C are subsets of U. B is the complement of A.

Appendix E

Diminished Scale Subsets



Appendix F

Twinkle, Twinkle Little Star for Saxophone Quartet

Music arranged, performed, and recorded by Ed Calle

TWINKLE, TWINKLE LITTLE SAXES


Twinkle, Twinkle Little Saxes
8
B
S. Sx.
A. Sx.
T. Sx.
B. Sx.
17
18
19
20
C
21
22
23
24


Twinkle, Twinkle Little Saxes
4
S. Sx.
A. Sx.
T. Sx.
B. Sx.
25
26
27
28
29
30
31
32
33

Appendix G

Audio Examples

All music featured in Appendix G was performed and recorded by Ed Calle

Recorded at One-Take Studios, Miami, FL

Twinkle, Twinkle Little Star (Arranged by Ed Calle)

Equal-tempered tuning version

Pythagorean tuning version

Doriana (Ed Calle, 2006) (36^{th} Street Music/BMI)

Equal-tempered tuning version

Pythagorean tuning version